LEMMA 6.1 *All right head moves or all left tape head moves*

If Turing Machine (Q, A, η) can execute $|Q|$ consecutive computational steps which are all right tape head moves or all left tape head moves without halting, then the machine has an immortal periodic point with period $\leq |Q|$.

PROOF. CONSECUTIVE RIGHT TAPE HEAD MOVES.

Suppose there are $|Q|$ consecutive computational steps which are all right tape head moves. Thus, there is a sequence of $|Q|$ commands $\eta(q_k, a_k) = (q_{k+1}, b_k, R)$ satisfying $1 \leq k \leq |Q|$ that execute these $|Q|$ computational steps without halting.

In the table below, the subscript k in each state q_k indicates the state of the machine just before the kth computational step.

STATE			k					$k+\|Q\|-1$	$k+\|Q\|$
q_1			$\underline{a}_1$	a_2	. . .	a_k	. . .	$a_{\|Q\|}$	
q_2			b_1	$\underline{a}_2$	. . .	a_k	. . .	$a_{\|Q\|}$	
. . .									
q_k			b_1	b_2	. . .	$\underline{a}_k$	. . .	$a_{\|Q\|}$	
. . .									
$q_{\|Q\|}$			b_1	b_2	. . .	b_k	. . .	$\underline{a}_{\|Q\|}$	
$q_{\|Q\|+1}$			b_1	b_2	. . .	b_k	. . .	$b_{\|Q\|}$	$\underline{a}_{\|Q\|+1}$

The Dirichlet Box principle applied to the $|Q| + 1$ states $\{q_1, q_2 \ldots q_{|Q|}, q_{|Q|+1}\}$ implies that two of the states are equal; namely, $q_j = q_k$ for some $j < k$. Thus, the point

$$p = \left[q_j, \overline{b_j b_{j+1} \ldots b_{k-1}}\ \overline{\langle a_j \rangle a_{j+1} \ldots a_{k-1}}\right]$$ is an immortal periodic point with period

$k - j$ and $(k - j) \leq |Q|$.

CONSECUTIVE LEFT TAPE HEAD MOVES. Suppose there are $|Q|$ consecutive computational steps which are all left tape head moves. Thus, there is a sequence of $|Q|$ commands $\eta(q_k, a_k) = (q_{k+1}, b_k, L)$ satisfying $1 \leq k \leq |Q|$ that execute these $|Q|$ computational steps without halting. In the table below, the subscript k in each state q_k indicates the state of the machine just before the kth computational step.

STATE		$k - \|Q\|$	$k - \|Q\| + 1$					k
q_1			$a_{\|Q\|}$	. . .	a_k	. . .	a_2	$\underline{a}_1$
q_2			$a_{\|Q\|}$	. . .	a_k	. . .	$\underline{a}_2$	b_1
. . .								
q_k			$a_{\|Q\|}$	. . .	$\underline{a}_k$		b_2	b_1
. . .								
$q_{\|Q\|}$			$\underline{a}_{\|Q\|}$	. . .	b_k	. . .	b_2	b_1
$q_{\|Q\|+1}$		$\underline{a}_{\|Q\|+1}$	$b_{\|Q\|}$	. . .	b_k	. . .	b_2	b_1

The Dirichlet Box principle applied to the $|Q| + 1$ states $\{q_1, q_2, \dots q_{|Q|}, q_{|Q|+1}\}$ implies that two of the states are equal; namely, $q_j = q_k$ for some $j < k$. Thus, the point $p = \left[q_j, \overline{a_{k-1} \dots \langle a_j \rangle}\ \overline{b_{k-1} \dots b_j}\right]$ is an immortal periodic point with period $k - j$ and $(k - j) \leq |Q|$.

REMARK 6.2

Consider Turing Machine (Q, A, η). If for some q in Q, and for some a, b in A, $\eta(q, a) = (q, b, R)$ or $\eta(q, a) = (q, b, L)$, then (Q, A, η) has an immortal fixed point.

PROOF. If $\eta(q, a) = (q, b, R)$, then $p = \left[q, \bar{b}\langle a\rangle\bar{a}\right]$ is an immortal fixed point of (Q, A, η). If $\eta(q, a) = (q, b, L)$, then $p = \left[q, \bar{a}\langle a\rangle\bar{b}\right]$ is an immortal fixed point of (Q, A, η).

DEFINITION 6.3 *Minimal Period*

If (Q, A, η) has no immortal orbits, then it is called a *halting machine*. Otherwise, the minimum$\{C(p) : p$ is an immortal periodic point of $(Q, A, \eta)\}$ is well-defined because it is the minimum of a non-empty subset of the natural numbers. This minimum is called the *minimal period* of (Q, A, η).

THEOREM 6.4 *Two State Minimal Period Theorem*

If $Q = \{q, r\}$ has only two non-halting states and $A = \{0, 1\}$, then (Q, A, η) is a halting machine or its minimal period is 1, 2 or 4.

PROOF. If (Q, A, η) is a halting machine, then the proof is completed. Otherwise, suppose (Q, A, η) has an immortal periodic point with period ≥ 5. Then it will be shown that this implies the machine must have an immortal periodic point with period ≤ 4.

Consider the first 5 computational steps of the immortal periodic point p. If two consecutive steps stay in the same state, then by remark 6.2, then (Q, A, η) has an immortal periodic point with period 1. Done. Furthermore, if there are two consecutive right tape head moves or left tape head moves, during these five computational steps, then by lemma 6.1, there is an immortal periodic point with period less than or equal to 2.

Thus, W.L.O.G. (due to symmetry), for the remaining cases the first five computational steps look like -- where the variables $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ represent elements of A:

STATE					MOVE	STEP
q		$\underline{x}_1$	y_1			
r		x_2	$\underline{y}_1$		R	1
q		$\underline{x}_2$	y_2		L	2
r		x_3	$\underline{y}_2$		R	3
q		$\underline{x}_3$	y_3		L	4
r		x_4	$\underline{y}_3$		R	5

OBSERVATION 1.

$x_1 = x_2$ implies $x_1 = x_2 = x_3$ because computational steps 1 and 3 are $\eta(q, x_1) = (\text{r}, x_2, R)$ and $\eta(q, x_2) = (\text{r}, x_3, R)$.

OBSERVATION 2.

$y_1 = y_2$ implies that $y_1 = y_2 = y_3$ because computational steps 2 and 4 are $\eta(\text{r}, y_1) = (q, y_2, L)$ and $\eta(\text{r}, y_2) = (q, y_3, L)$

Since A has 2 elements, $[x_1 = x_2$ or $x_1 = x_3$ or $x_2 = x_3]$ and $[y_1 = y_2$ or $y_1 = y_3$ or $y_2 = y_3]$.

CASE 1: $x_2 = x_3$ and $y_2 = y_3$. Based on steps 3 and 4, point $p = \left[q, \bar{0}\langle x_2 \rangle y_2 \bar{0}\right]$ is an immortal periodic point with period 2.

Many of the nine cases below are reduced to previous cases.

CASE 2: $x_1 = x_2$ and $y_2 = y_3$. The first observation reduces case 2 to case 1.

CASE 3: $x_1 = x_3$ and $y_2 = y_3$. By replacing all occurrences of x_3 by x_1 and all occurrences of y_3 by y_2, then the previous table becomes:

STATE					MOVE	STEP
q		$\underline{x_1}$	y_1			
r		x_2	$\underline{y_1}$		R	1
q		$\underline{x_2}$	y_2		L	2
r		x_1	$\underline{y_2}$		R	3
q		$\underline{x_1}$	y_2		L	4
r		x_2	$\underline{y_2}$		R	5
q		$\underline{x_2}$	y_2		L	6

After the substitution, from step 4, then $\eta(r, y_2) = (q, y_2, L)$. This implies step 6 in the table.

Looking at steps 2 and 6, point $p = \left[q, \bar{0}\langle x_2\rangle y_2\bar{0}\right]$ is an immortal periodic point with period 2 or 4.

CASE 4: $x_1 = x_3$ and $y_1 = y_3$. Substituting x_1 for x_3 and y_1 for y_3 in step 4 of the original table, then the point $p = \left[q, \bar{0}\langle x_1\rangle y_1\bar{0}\right]$ is an immortal periodic point with period 2 or 4.

CASE 5: $x_1 = x_2$ and $y_1 = y_3$. This reduces to case 4 from the first observation.

CASE 6: $x_2 = x_3$ and $y_1 = y_3$. Substituting x_2 for x_3 and y_1 for y_3 in the original table and observing that from step 3 that $\eta(q, x_2) = (q, x_2, R)$. This implies that $x_4 = x_2$

STATE					MOVE	STEP
q		$\underline{x_1}$	y_1			
r		x_2	$\underline{y_1}$		R	1
q		$\underline{x_2}$	y_2		L	2
r		x_2	$\underline{y_2}$		R	3
q		$\underline{x_2}$	y_1		L	4
r		x_2	$\underline{y_1}$		R	5

Then observe that after step 1 and step 5, the points are identical. Thus, the point

$p = \left[r, \bar{0}\, x_2 \langle y_1 \rangle\, \bar{0}\right]$ is immortal with period equal to 2 or 4.

CASE 7: $x_1 = x_2$ and $y_1 = y_2$. This reduces to case 2 from the second observation.

CASE 8: $x_1 = x_3$ and $y_1 = y_2$. This reduces to case 4 from the second observation.

CASE 9: $x_2 = x_3$ and $y_1 = y_2$. This reduces to case 6 from the second observation.

Finally, it is shown that any machine having an immortal periodic point with period 3 must have an immortal periodic point with period 1 or 2. Suppose the machine has an immortal period 3 point. During the three computational steps, the claim is that there has to be two consecutive steps that are in the same state. For example, the state sequence q, r, q, r would contradict that it is a period 3 orbit because at step 0 it is in state q and after step 3 it is in state r; similarly, the state sequence r, q, r, q would contradict that is a period 3 orbit. Thus, there must be two consecutive steps that are in the same state, which implies it is an immortal fixed point. Thus, the machine can not have a minimal period of 3.